UDC 62-50

THE DIFFERENTIAL GAME OF EVASION IN A PLANE

PMM Vol. 41, №4, 1977, pp. 604-608

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(Received December 21, 1976)

Necessary and sufficient conditions of point avoidance in a strictly linear differential game in a plane are presented. This paper is related to [1-4].

1. Let the motion of a conflict-controlled system in the Euclidean plane X be defined by the differential equation

$$dx/dt = Ax + f(u, v) \tag{1.1}$$

where x is a two-dimensional phase vector, A is a constant 2×2 matrix, f is a continuous function with values in X and specified in compactum G belonging to the product $X_u \times X_v$ of finite-dimensional Euclidean spaces. The selection of controls u and v is effected by the first and second player, respectively.

We denote by P (Q) the orthogonal projection of G on X_u (X_v), and set $P(v) = \{u \in P : (u, v) \in G\}$, $v \in Q$ $(Q(u) = \{v \in Q : (u, v) \in G\}, u \in P)$. We assume that $P(v)$ $(Q(u))$ depends on v (u) in the sense of Hausdorff's metric.

Let $\mathbf{U}$ $(\mathbf{V})$ be the set of strategies of the first (second) player, namely set of all functions determined in $\overline{R_+} \times X$ with values in P (Q), where R_+ is a set of positive numbers and the vinculum denotes closure. We denote by $\mathbf{U}^v$ $(\mathbf{V}^u)$ the set of all functions measurable in t for any $v \in Q$ $(u \in P)$, which associate to every vector (t, v) $((t, u))$ in $\overline{R_+} \times Q$ $(\overline{R_+} \times P)$ a vector in $P(v)$ $(Q(u))$.

Let Δ be an arbitrary subdivision of the semiaxis $\overline{R_+}$ by points $0 = t_1 < t_2 < \ldots$, $\lim t_i = \infty$ when $i \to \infty$. We denote by $d(\Delta)$ the diameter of subdivision Δ i.e. $\sup\{|t_{i+1} - t_i| : i = 1, 2, \ldots\}$, and for fixed Δ, $y \in X$, $U \in \mathbf{U}$ $(V \in \mathbf{V})$, and $V^u \in \mathbf{V}^u$ $(U^v \in \mathbf{U}^v)$ we use symbol $x(\cdot; \Delta, y, U, V^u)$ $(x(\cdot; \Delta, y, U^v, V))$ for denoting an absolutely continuous function specified in $\overline{R_+}$ with values in X, equal y at $t = 0$, and in every half-open interval $t_i \leqslant t < t_{i+1}$, $i = 1, 2, \ldots$ of subdivision Δ is the solution of the differential equation

$$dx/dt = Ax + f(U(t_i, x(t_i)), \quad V^u(t, U(t_i, x(t_i))))$$
$$(dx/dt = Ax + f(U^v(t, V(t_i, x(t_i)), V(t_i, x(t_i))))$$

Let m denote the coordinate origin and $O(\varepsilon, x)$ denote the ε-neighborhood of point $x \in X$. We introduce sets B_1 and B_2.

The set B_1 is the totality of all points $y \in X$ for each of which it is possible to select a strategy $U \in \mathbf{U}$, instant $\Theta \geqslant 0$, and the mapping $\varepsilon \to \delta(\varepsilon)$ from R_+ into R_+ so that for any $\varepsilon > 0$ the subdivision Δ of diameter $d(\Delta) \leqslant \delta(\varepsilon)$ and function $V^u \in \mathbf{V}^u$ at some $t \in [0, \Theta]$ the inclusion $x(t; \Delta, y, U, V^u) \in O(\varepsilon, m)$ is satisfied.

The set B_2 is the totality of all points $y \in X$ for each of which it is possible to select a strategy $V \in \mathbf{V}$ and mapping of $\Theta \to \varepsilon(\Theta)$ and $\Theta \to \delta(\Theta)$ from R_+ into R_+ so that for any $\Theta > 0$ the subdivision Δ of diameter $d(\Delta) \leqslant \delta(\varepsilon)$ and function $U^v \in \mathbf{U}^v$ the inclusion $x(t; \Delta, y, U^v, V) \in X \setminus O(\varepsilon, m)$ is satisfied for any $t \in [0, \Theta]$.

In other words the set B_1 (B_2) is the totality of all initial points y in plane X for each of which there exists a method of action of the first (second) player that makes it possible for him to bring system (1.1) fairly close to the terminal point m (makes it possible to prevent system (1.1) from reaching point m in any finite time) for any actions of the second (first) player.

Below we present the necessary and sufficient conditions for $B_1 \neq \{m\}$ $(B_2 = X \setminus \{m\})$.

2. We denote by Γ the set of all functions that associate to every vector u in P a vector in $Q(u)$. Let

$$H_1(x, \gamma) = \mathrm{co} \bigcup_{u \in P} [-Ax - f(u, \gamma(u))], \quad x \in X, \quad \gamma \in \Gamma$$
$$H_2(x, v) = \mathrm{co} \bigcup_{u \in P(v)} [-Ax - f(u, v)], \quad x \in X, \quad v \in Q$$

where $\mathrm{co}\, D$ is the closed convex envelope of set D. For any arbitrary convex closed set $D \subset X$ we assume

$$\Lambda(D) = \{x : x = \lambda z, \quad z \in D, \quad \lambda > 0\}$$
$$D^\circ = \overline{\Lambda}(D) \cap \{x : |x| = 1\}$$

Assuming everywhere below $\xi = 1, 2$, we denote $W_\xi = \Gamma$ for $\xi = 1$ and $W_\xi = Q$ when $\xi = 2$. Let for any $x \in X$

$$K_\xi(x) = \bigcap_{w \in W_\xi} H_\xi^\circ(x, w) \tag{2.1}$$

$L_\xi(x) = K_\xi(x)$, if $K_\xi(x) \neq \varnothing$ and consists of a single element; in the opposite case $L_\xi(x) = \varnothing$.

Assumption 1. If $K_\xi(m) \neq \varnothing$ and consists of one or two elements, then

$$\inf_{p, w} \max \{\lambda \geqslant 0 : \lambda p \in H_\xi(m, w)\} > 0$$

where the exact lower bound is taken over all $p \in K_\xi(m)$, $w \in W_\xi$.

Before formulating the second assumption, we introduce the following concepts. For $L_\xi(m) \neq \varnothing$ we set $F_\xi = \{x : x = \lambda L_\xi(m), \lambda \in R\}$ where R is a set of real numbers. When the straight line F_ξ is not invariant with respect to linear transformation defined by matrix A, then we assume that p_ξ is a unit vector that satisfies conditions $p_\xi' L_\xi(m) = 0$ and $p_\xi' A L_\xi(m) > 0$ where the prime indicates transposition. For any $c > 0$ we assume that

$$J_\xi^{1,c} = \{l \in X : l' p_\xi \geqslant 0, \quad c\,|l| \geqslant l' L_\xi(m) \geqslant 0\},$$
$$J_\xi^{2,c} = -J_\xi^{1,c}$$

and for any $l \in X$

$$S_1(l) = \max_{u \in P} \min_{v \in Q(u)} l' f(u, v), \quad S_2(l) = \min_{v \in Q} \max_{u \in P(v)} l' f(u, v)$$

Assumption 2. If $L_\xi(m) \neq \varnothing$ and the straight line F_ξ are not invariant, there exists such $\alpha > 0$ for which function S_ξ is either convex in each of the sets $J_\xi^{1,\alpha}$ and $J_\xi^{2,\alpha}$ or is concave on each of these sets.

We set $E_1 = B_1$ and $E_2 = X \setminus B_2$.

Theorem. Let Assumptions 1 and 2 be satisfied. For $E_\xi \neq \{m\}$ it is necessary and sufficient if one of the following two conditions is satisfied:

1) $K_\xi(m) \neq \varnothing$, $L_\xi(m) = \varnothing$.

2) $L_\xi(m) \neq \varnothing$ and there exist a $\varkappa > 0$ such that $K_\xi(x) \neq \varnothing$ for any $x \in \Lambda(L_\xi(m)) \cap O(\varkappa, m)$.

Notes. 1°. An equivalent definition of the set $K_\xi(x)$ introduced by formula (2.1) can be derived as follows. Let $\nu_\xi(x)$ be the totality of all unit vectors l such that $S_\xi(l) + l\,Ax \leqslant 0$. Then

$$K_\xi(x) = \bigcap_{l \in \nu_\xi(x)} \{z : |z| = 1,\ l'z \geqslant 0\}$$

if $\nu_\xi(x) \neq \varnothing$, and $K_\xi(x) = \{z : |z| = 1\}$ when $\nu_\xi(x) = \varnothing$.

2°. If for any $l \in X$

$$S_1(l) = S_2(l) \qquad (2.2)$$

(i.e. the condition of saddle point is satisfied in the small game [1]), then $K_1(x) = K_2(x)$ for any $x \in X$. When (2.2) holds and the set $K_1(m) = K_2(m)$ consists of one or two elements, the fulfilment of Assumption 1 for $\xi = 1$ entails its fulfilment for $\xi = 2$ and vice versa.

3°. Assumption 1 is satisfied if, for instance,

$$f(u, v) = u - v, \quad G = P \times Q, \quad P \subset X, \quad Q \subset X \qquad (2.3)$$

and the set co P is a polygon. Assumption 2 is satisfied if, for instance, at least one of sets co P or co Q is a polygon.

3. Let us outline the proof of the theorem. Let $K_\xi(x) \neq \varnothing$ for some $x \in X$ and $r_\xi(x)$ be some arbitrary vector in $K_\xi(x)$. We set

$$\eta(x, r_\xi(x)) = \inf_{w \in W_\xi} \max\{\lambda \geqslant 0 : \lambda r_\xi(x) \in H_\xi(x, w)\}$$

We denote by Π that of the two closed half-planes determined by the straight line $\{x : Ax \in F_\xi\}$ whereinto is directed vector $L_\xi(m)$, when $L_\xi(m) \neq \varnothing$ and the straight line F_ξ is not invariant. We assume that $\Pi(c) = O(c, m) \cap \Pi$, $c > 0$. The following lemma is valid.

Lemma 1. If $L_\xi(m) \neq \varnothing$, F_ξ not invariant, and the Assumptions 1 and 2 are satisfied, then, either a) there exist a $\varkappa > 0$ and function q_ξ that satisfies the Lipschitz condition and is determined in $O(\varkappa, m)$ with values in X°, such that $K_\xi(x) \neq \varnothing$ and $q_\xi(x) \in K_\xi(x)$ for any $x \in \Pi(\varkappa)$ and $\inf\{\eta(x, q_\xi(x)) : x \in \Pi(\varkappa)\} > 0$, or b) there exist such $\varkappa > 0$ and functions h_ξ and ψ_ξ that satisfy the Lipschitz condition and are determined in $O(\varkappa, m)$ with values in $J_\xi^{\circ 1,\alpha}$ and R_+, respectively, such that

$$\max_{w \in W_\xi} \min \{(-1)^i h_\xi'(x)\, y:\ \ y \in H_\xi(x, w)\} \geqslant \psi_\xi(x)$$

$$i = 1, 2; \quad x \in \Pi(\varkappa)$$

$$h_\xi(x) \neq p_\xi, \quad \psi_\xi(x) > 0, \quad x \in \Pi(\varkappa) \setminus F_\xi$$

$$h_\xi(x) = p_\xi, \quad \psi_\xi(x) = 0, \quad x \in \Pi(\varkappa) \cap F_\xi$$

Depending on the particular form of system (1.1) and the fulfilment of Assumptions 1 and 2 only one of the following five cases is possible:

1) $K_\xi(m) \neq \varnothing, \quad L_\xi(m) = \varnothing$;
2) $L_\xi(m) \neq \varnothing$, F_ξ is invariant;
3) $L_\xi(m) \neq \varnothing$, F_ξ is not invariant and statement a) of Lemma 1 is satisfied;
4) $L_\xi(m) \neq \varnothing$, F_ξ is not invariant and statement b) of Lemma 1 is satisfied, and
5) $K_\xi(m) = \varnothing$.

Lemma 2. If assumptions 1 and 2 are satisfied, then $E_\xi \neq \{m\}$ in cases 1-3, and $E_\xi = \{m\}$ in cases 4 and 5.

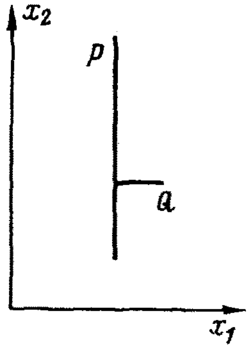

Fig. 1

The theorem follows from Lemma 2 if one takes into consideration the following observations:

statement a) of Lemma 1 implies the fulfilment of condition 2) of the theorem;

if statement b) of Lemma 1 is valid condition 2) of the theorem is not satisfied;

condition 2) of the theorem is satisfied, when $L_\xi(m) \neq \varnothing$ and the straight line F_ξ is invariant.

4. Examples. Let function f and set G be of the form (2.3). If the sets P and Q are such as shown in Figs. 1 and 2, $K_1(m) = K_2(m) = \varnothing$, hence for any matrix A we have, according to the theorem, $B_1 = \{m\}$, $B_2 = X \setminus \{m\}$. Now, let the sets P and Q be such as shown in Fig. 3, then $L_1(m) = L_2(m) = \{l : l_1 = 1, l_2 = 0\}$. If

$$A = \begin{Vmatrix} 0 & 0 \\ -1 & 0 \end{Vmatrix} \quad \left(A = \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}\right)$$

then there exists such $\varkappa > 0$ that $K_1(x) = K_2(x) = \varnothing$ ($K_1(x) = K_2(x) \neq \varnothing$) for any $x \in \{z : 0 < z_1 < \varkappa,\ z_2 = 0\}$. Hence $B_1 = \{m\}$, $B_2 = X \setminus \{m\}$ ($B_1 \neq \{m\}$, $B_2 \neq X \setminus \{m\}$).

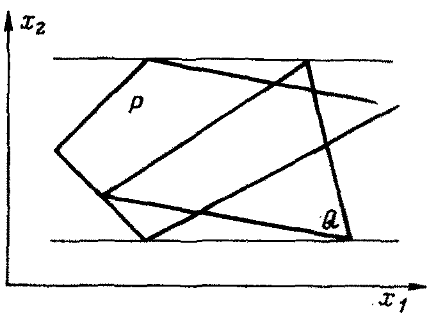

Fig. 2

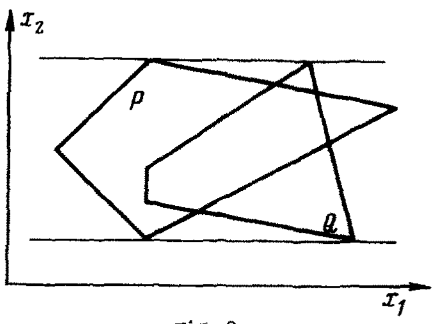

Fig. 3

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Translated by J.J. D.